# 6-2 Capacitance

**Reading Assignment:** *pp. 179-185*

To **create** a potential difference $V$ between two perfect conductors, we must **deposit** charge Q on the conductor surface.

Or, if we **enforce** a potential difference $V$ (with a voltage source) between the two perfect conductors, some charge Q will be **stored** on the conductor surface.

**Q:**

**A:**

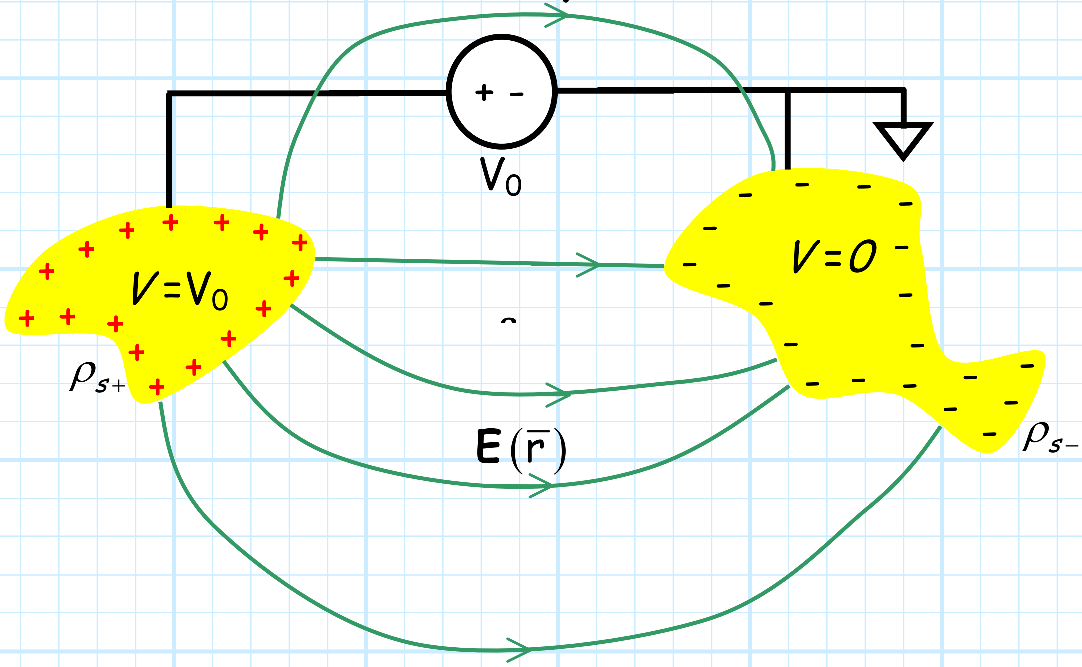
**HO: Capacitance**

**HO: The Parallel Plate Capacitor**

**HO: Capacitance of a Coaxial Transmission Line**

# Capacitance

Consider two **conductors**, with a **potential difference** of V volts.




* Since there is a potential difference between the conductors, there must be an **electric potential field** $V(\bar{r})$, and therefore an **electric field** $\mathbf{E}(\bar{r})$ in the region between the conductors.

* Likewise, if there is an electric field, then we can specify an **electric flux density** $\mathbf{D}(\bar{r})$, which we can use to determine the **surface charge density** $\rho_s(\bar{r})$ on each of the conductors.

* We find that if the total net charge on **one** conductor is Q then the charge on the **other** will be equal to –Q.

In other words, the total net charge on each conductor will be **equal** but **opposite**!

Note that this does **not** mean that the surface charge densities on each conductor are equal (i.e., $\rho_{s+}(\bar{r}) \neq \rho_{s-}(\bar{r})$). Rather, it means that:

$$\oiint_{S_+} \rho_{s+}(\bar{r})\, ds = -\oiint_{S_-} \rho_{s-}(\bar{r})\, ds = Q$$

where surface $S_+$ is the surface surrounding the conductor with the positive charge (and the higher electric potential), while the surface $S_-$ surrounds the conductor with the negative charge.

**Q:** *How much free **charge** Q is there on each conductor, and how does this charge relate to the **voltage** $V_0$?*

**A:** We can determine this from the mutual **capacitance** $C$ of these conductors!

The mutual **capacitance** between two conductors is **defined** as:

$$C = \frac{Q}{V} \qquad \left[\frac{\text{Coulombs}}{\text{Volt}} \doteq \text{Farad}\right]$$

where Q is the **total charge** on **each conductor**, and $V$ is the **potential difference** between each conductor (for our example, $V = V_0$).

Recall that the total charge on a conductor can be determined by **integrating** the surface charge density $\rho_s(\bar{r})$ across the **entire surface** S of a conductor:

$$Q = \oiint_{S_+} \rho_{s+}(\bar{r})\, ds = -\oiint_{S_-} \rho_{s-}(\bar{r})\, ds$$

But recall also that the surface charge density on the surface of a conductor can be determined from the **electric flux density** $\mathbf{D}(\bar{r})$:

$$\rho_s(\bar{r}) = \mathbf{D}(\bar{r}) \cdot \hat{a}_n$$

where $\hat{a}_n$ is a unit vector **normal** to the conductor.

**Combining** the two equations above, we get:

$$Q = \oiint_{S_+} \mathbf{D}(\bar{r}) \cdot \hat{a}_n ds = -\oiint_{S_-} \mathbf{D}(\bar{r}) \cdot \hat{a}_n ds$$

$$= \oiint_{S_+} \mathbf{D}(\bar{r}) \cdot \overline{ds} = -\oiint_{S_-} \mathbf{D}(\bar{r}) \cdot \overline{ds}$$

where we remember that $\overline{ds} = \hat{a}_n\, ds$ .

Hey! This is **no surprise**! We **already** knew that:

$$Q = \oiint_{S} \mathbf{D}(\bar{r}) \cdot \overline{ds}$$

This expression is also know as___________________!!

Note since $\mathbf{D}(\bar{r}) = \varepsilon \mathbf{E}(\bar{r})$ we can also say:

$$Q = \oiint_{S_+} \varepsilon \mathbf{E}(\bar{r}) \cdot \overline{ds}$$

The **potential difference** $V$ between two conductors can likewise be determined as:

$$V = \int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell}$$

where $C$ is **any contour** that leads from one conductor to the other.

**Q:** Why **any** contour?

**A:**

We can therefore determine the **capacitance** between two conductors as:

$$C = \frac{Q}{V} = \frac{\oiint_{S_+} \varepsilon \mathbf{E}(\bar{r}) \cdot \overline{ds}}{\int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell}} \quad [\text{Farad}]$$

Where the contour $C$ must start at **some** point on surface $S_+$ and end at **some** point on surface $S_-$ .

Note this expression can be written as:

$$Q = C\,V$$

In other words, the charge **stored** by two conductors is equal to the product of their mutual capacitance and the potential difference between them.

Therefore, the **greater** capacitance, the **greater** the amount of **charge** that is stored.

By the way, try taking the **time derivative** of the above equation:

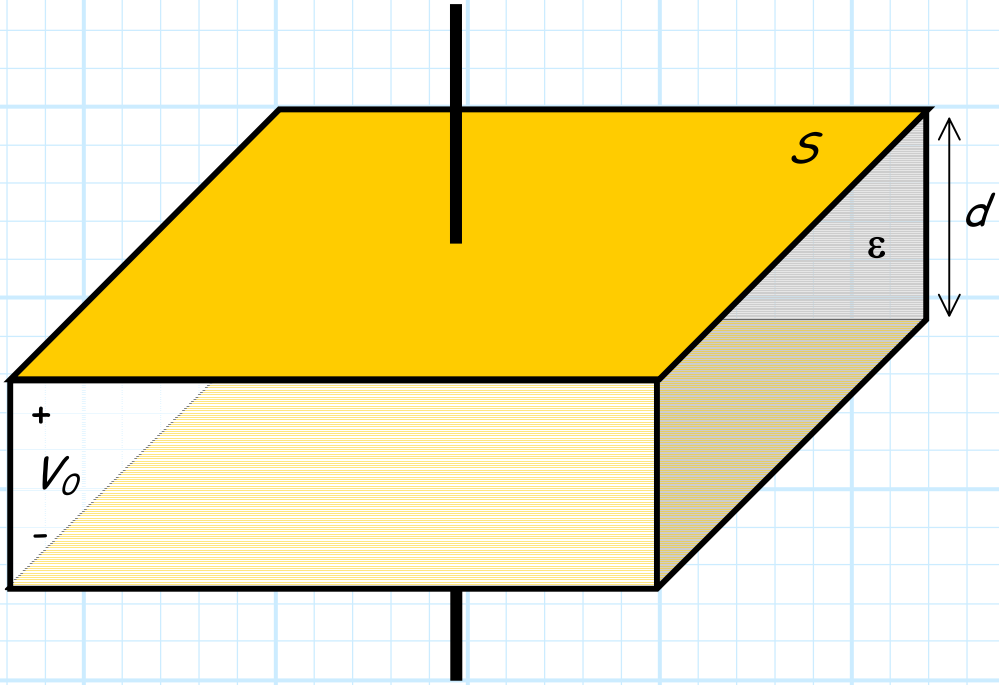
$$\frac{dQ}{dt} = C\,\frac{dV}{dt}$$
$$I = C\,\frac{dV}{dt}$$

Look familiar ?

By the way, the current $I$ in this equation is **displacement current**.

# The Parallel Plate Capacitor

Consider the geometry of a **parallel plate capacitor**:




Where:

$V_0$ = the **potential difference** between the plates
$S$ = **surface area** of each conducting plate
$d$ = **distance** between plates
$\varepsilon$ = **permittivity** of the dielectric between the plates

Recall that we determined the fields and surface charge density of an **infinite** pair of parallel plates. We can use those results to approximate the fields and charge densities of this **finite** structure, where the **area** of each plate is $S$.

For example, we determined that the **surface charge density** on the upper plate is:

$$\rho_{s+}(\bar{r}) = \frac{\varepsilon V_0}{d}$$

The **total charge** on the upper plate is therefore:

$$\begin{aligned} Q &= \iint_{S_+} \rho_{s+}(\bar{r})\, ds \\ &= \iint_{S_+} \frac{\varepsilon V_0}{d}\, ds \\ &= \frac{\varepsilon V_0}{d} \iint_{S_+} ds \\ &= \frac{\varepsilon V_0 S}{d} \end{aligned}$$

The **capacitance** of this structure is therefore:

$$C = \frac{Q}{V} = \left(\frac{\varepsilon V_0 S}{d}\right)\left(\frac{1}{V_0}\right) = \frac{\varepsilon S}{d} \quad [\text{Farads}]$$

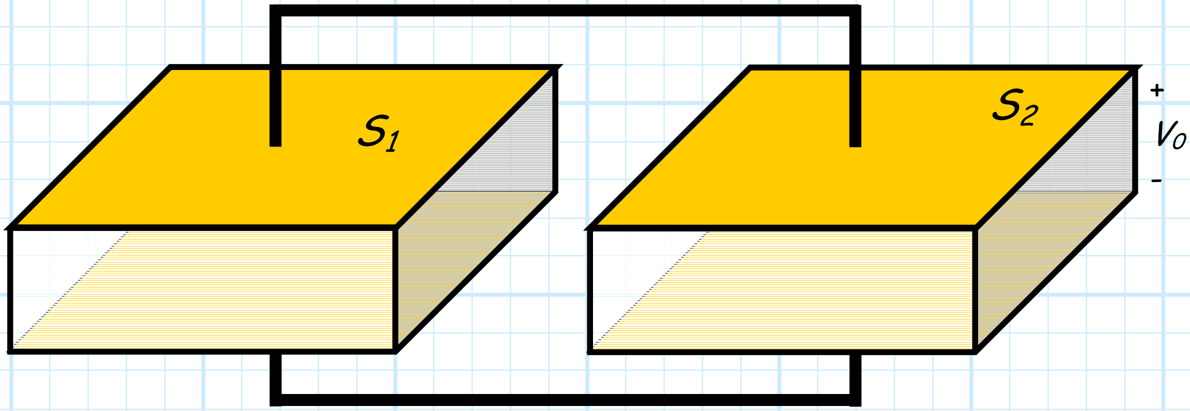
Note therefore, that we can **increase** the capacitance of a parallel plate capacitor by:

**1)** **Increasing** surface area *S*.

**2)** Decreasing separation distance *d*.

**3)** **Increasing** the dielectric permittivity $\varepsilon$.

Consider now the structure:




Note the **two** upper plates form **one** conducting structure, and the **two** bottom plates form **another**.

**Q:** *What is the **capacitance** between these two conducting structures?*

**A:** The potential difference between them is $V_0$. The **total charge** on one conducting structure is simply the **sum** of the charges on **each plate**:

$$Q = Q_1 + Q_2 = \frac{\varepsilon V_0 S_1}{d} + \frac{\varepsilon V_0 S_2}{d}$$
$$= \frac{\varepsilon V_0 \left(S_1 + S_2\right)}{d}$$

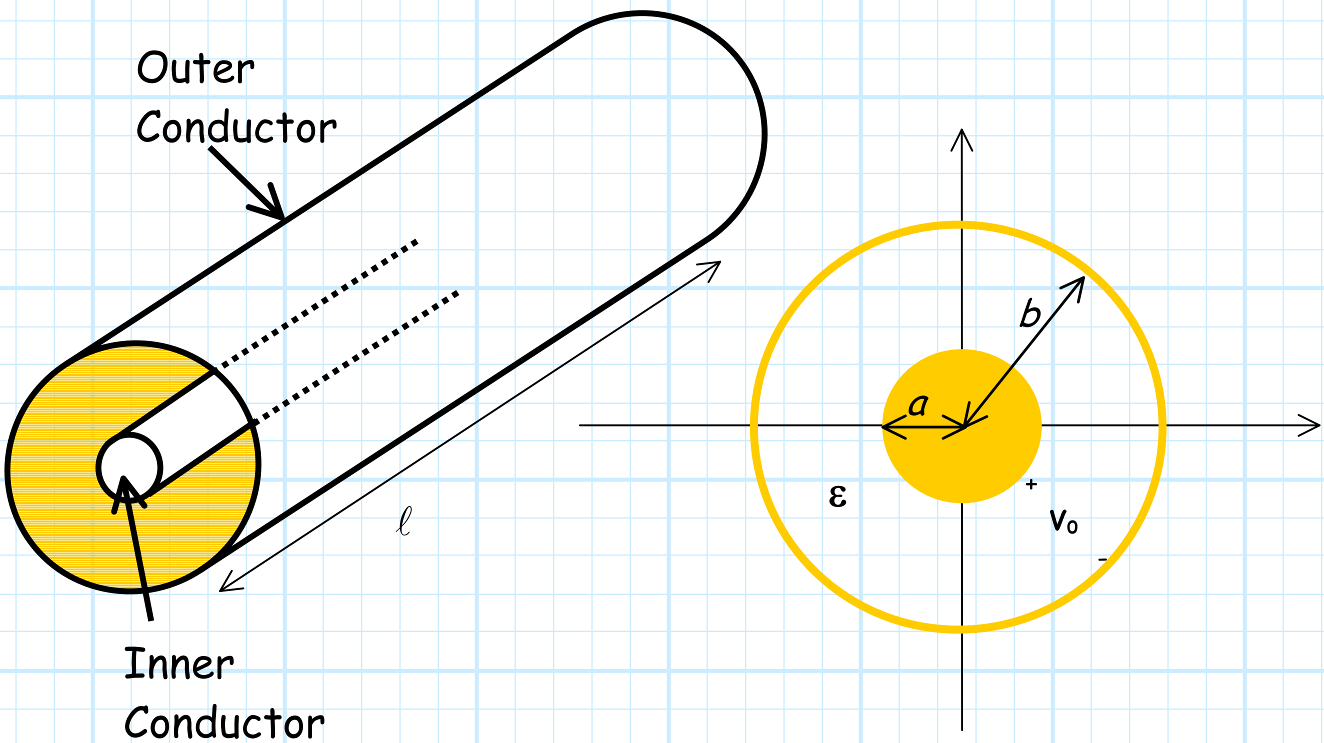
Therefore, the **capacitance** of this structure is:

$$\begin{aligned} C = \frac{Q}{V} &= \left(\frac{\varepsilon V_0 \left(S_1 + S_2\right)}{d}\right)\left(\frac{1}{V_0}\right) \\ &= \frac{\varepsilon\left(S_1 + S_2\right)}{d} \\ &= \frac{\varepsilon S_1}{d} + \frac{\varepsilon S_2}{d} \\ &= C_1 + C_2 \end{aligned}$$

But **you** knew this! The total capacitance of two capacitors in **parallel** is equal to the **sum** of **each** capacitance.

# Capacitance of a Coaxial Transmission Line

Recall the geometry of a **coaxial transmission line**:




Coax Cross-Section

We earlier determined that if a **potential difference** of $V_0$ volts is placed across the conductors, the **surface charge density** on the **inner** conductor is:

$$\rho_{sa}(\bar{r}) = \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \qquad (\rho = a)$$

The **total charge** $Q$ on the **inner** conductor of a coax of length $\ell$ is determined by **integrating** the surface charge density across the **conductor surface**:

$$
\begin{aligned}
Q &= \oiint_{S_+} \rho_{sa}(\bar{r})\, ds \\
&= \int_0^{\ell}\int_0^{2\pi} \frac{\varepsilon V_0}{\ln[b/a]}\frac{1}{a}\;\rho\, d\phi\, dz \\
&= \frac{\varepsilon V_0}{\ln[b/a]}\frac{1}{a}\int_0^{\ell}\int_0^{2\pi} a\, d\phi\, dz \\
&= \frac{\varepsilon V_0}{\ln[b/a]}\frac{1}{a}\;\ell(2\pi a) \\
&= V_0 \frac{2\pi\varepsilon}{\ln[b/a]}\ell
\end{aligned}
$$

Note since $\rho_{sa}(\bar{r}) = \mathbf{D}(\bar{r})\cdot\hat{a}_n$, we would have arrived at the **same** result by using:

$$
Q = \oiint_{S_+} \varepsilon\mathbf{E}(\bar{r})\cdot\overline{ds}
$$

We can now determine the **capacitance** of this coaxial line!

Since $C = Q/V$, and since the **potential difference** between the conductors is $V = V_0$, we find:

$$
\begin{aligned}
C = \frac{Q}{V} &= \left(V_0\frac{2\pi\varepsilon}{\ln[b/a]}\ell\right)\left(\frac{1}{V_0}\right) \\
&= \frac{2\pi\varepsilon}{\ln[b/a]}\ell
\end{aligned}
$$

This value represents the capacitance of a coaxial line of length $\ell$. A more useful expression is the capacitance of a coaxial line **per unit length** (e.g. farads/meter). We find this simply by **dividing** by length $\ell$:

$$\frac{C}{\ell} = \frac{2\pi\varepsilon}{\ln[b/a]} \qquad \left[\frac{\text{farads}}{\text{meter}}\right]$$

Note the **longer** the transmission line, the **greater** the capacitance!

This can cause **great difficulty** if the voltage across the transmission line conductors is **time varying** (as it almost certainly will be!).

For **long** transmission lines, engineers cannot consider a transmission line simply as a "**wire**" conductor that connects circuit elements together. Instead, capacitance (and inductance) make the transmission line **itself** a **circuit element**!

In this case, engineers must use **transmission line theory** to analyze circuits !